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Falkner–Skan flows past moving boundaries: an exactly solvable case

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Abstract The Falkner–Skan flows past stretching boundaries are revisited in this paper. The usual assumption $U_w(x) = \lambda\, U(x)$, i.e. the proportionality of the stretching velocity $U_w(x)$ and the free stream velocity $U(x)$ is adopted. For the special case of a converging channel (*wedge nozzle*), $U(x) \sim -1/x$, exact analytical solutions in terms of elementary hyperbolic functions are reported. In the range $-2 < \lambda < +1$ dual solutions describing either opposing ($\lambda < 0$) or aiding ($\lambda > 0$) flow regimes were found. In the range $\lambda > 1$ unique solutions occur, while below the critical value $\lambda_c = -2$ no solutions exist at all. The mechanical features of these solutions are discussed in some detail.

1 Introduction

After the publication of Heinrich Blasius' pioneering work in 1908, more than three decades elapsed until the uniqueness of Blasius' famous velocity boundary layer solution was rigorously proved by Weyl [30]. On this background it was quite surprising as, further three decades later, Klemp and Acrivos [13,14] reported that in the Blasius-problem non-unique solutions may occur when the plate is not at rest, but moves with a constant velocity U_w, opposite in direction to the free stream of velocity U (i.e. when $\text{sgn}\, U_w = -\text{sgn} U$). More precisely, for negative values of the velocity ratio $\lambda = U_w/U$ dual solutions exist as long as λ is larger than the critical value $\lambda_c = -0.3541$, below which no solutions exist. For $\lambda > 0$ (i.e. for $\text{sgn} U_w = \text{sgn} U$) on the other hand, Callegari and Nachman [7] have found unique solutions. Following the work of Klemp and Acrivos [14], various mathematical and physical features of multiple (velocity and temperature boundary-layer) solutions occurring under the simultaneous effect of a driving free stream and a moving (stretching) wall have been investigated in some detail by several further authors [1,5,10,11,15,24].

In the latter few years a renewed research interest can be observed for such "doubly driven" self-similar boundary layer flows [3,4,26]. Following the approach of Riley and Weidman [24], in all the extensions to the case of Falkner–Skan flows [8] past stretching boundaries it has been assumed that the ratio $\lambda = U_w(x)/U(x)$ of the non-uniform free stream velocity $U(x)$ and the wall velocity $U_w(x)$ is a constant. Concerning the occurrence of multiple solutions, both the *aiding* ($\lambda > 0$) and *opposing* ($\lambda < 0$) regimes of such flows are of physical interest. In this respect a special attention has recently been given to the stagnation point flows [12,17,19]). The additional effect of lateral mass flux on multiple solutions has been considered in this context by Weidman et al. [28,29]. It is also worth mentioning here that parallel with these studies of doubly driven flows of clear viscous fluids, the mathematically related problem of the solution non-uniqueness for flows in fluid saturated porous media has also intensively been investigated in the pertinent literature [18,19,21,23].

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The aim of the present paper is to contribute to the exploration of the solution space of *Falkner–Skan flows past stretching boundaries* with a further specific example. This is the case of the boundary layer flow in a converging channel (*wedge nozzle*), driven simultaneously by a free stream of velocity $U(x) = -U_0/(x/L)$, $U_0 > 0$ and boundaries stretching with the velocity $U_w(x) = \lambda\, U(x)$. The attractive feature of this case is that all the (unique and dual) solutions can be obtained in closed analytical form in terms of elementary hyperbolic functions. The domain of existence of these solutions extends to all values of the velocity ratio λ above the critical value $\lambda_c = -2$. In the range $-2 < \lambda < +1$ dual solutions describing either opposing ($\lambda < 0$) or aiding ($\lambda > 0$) flow regimes occur. For $\lambda = 0$, the famous Pohlhausen solution [22] and its dual counterpart are recovered, while the solutions associated with positive values of λ are proven to be unique.

2 Governing equations

We start our considerations with the general case of a power law free stream velocity

$$U(x) = s\,|U_\infty| \left(\frac{x}{L}\right)^m, \quad s = \mathrm{sgn}\,(U_\infty) = \pm 1, \tag{1}$$

where L is a reference length and the wall coordinate x is measured from the line of intersection of the two plane boundaries of the wedge, representing the leading edge of the flow. The sign s is $+1$ when the free stream velocity U is directed from the leading edge to plus infinity and -1 when it points in the opposite direction. According to Goldstein [9], in the latter case we are faced with a *backward* boundary layer flow, while for $s = +1$, the *forward*, or "usual" boundary layers occur. The stretching velocity $U_w(x)$ of the boundaries is assumed to be proportional to $U(x)$,

$$U_w(x) = \lambda\, U(x), \tag{2}$$

where the velocity ratio λ may take positive (aiding flows) or negative values (opposing flows), respectively. The value $\lambda = 0$ corresponds to the proper case of the classical Falkner–Skan flows over resting surfaces. The governing continuity and momentum balance equation in the boundary layer approximation are

$$\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} = 0, \tag{3}$$

$$u\frac{\partial u}{\partial x} + \upsilon\frac{\partial u}{\partial y} = \upsilon\frac{\partial^2 u}{\partial y^2} + U\frac{\mathrm{d}U}{\mathrm{d}x}. \tag{4}$$

The boundary conditions for the wedge flows driven simultaneously by a free stream and by impermeable stretching walls are

$$u(x,0) = \lambda U(x), \quad \upsilon(x,0) = 0, \quad u(x,\infty) = U(x). \tag{5}$$

Introducing the stream function $\psi\,(x,y)$ by the usual definition $u = \partial\psi/\partial y$, $\upsilon = -\partial\psi/\partial x$, as well as the similarity transformations

$$\psi\,(x,y) = s\,\upsilon\sqrt{Re}\left(\frac{x}{L}\right)^{\frac{m+1}{2}} f(\eta), \quad \eta = \sqrt{Re}\left(\frac{x}{L}\right)^{\frac{m-1}{2}}\frac{y}{L}, \quad Re = \frac{|U_\infty|\,L}{\upsilon}, \tag{6}$$

the boundary value problem (3)–(5) reduces to

$$s\,f''' + \frac{m+1}{2} f f'' + m\left(1 - f'^2\right) = 0, \tag{7}$$

$$f(0) = 0, \quad f'(0) = \lambda, \quad f'(\infty) = 1. \tag{8}$$

In the above equations the primes denote differentiations with respect to η and *Re* is the Reynolds number. The dimensional velocity field is obtained as

$$\begin{aligned} u\,(x,y) &= U(x)\,f'(\eta), \\ v\,(x,y) &= -\frac{U(x)}{\sqrt{Re}}\left(\frac{x}{L}\right)^{-\frac{m+1}{2}}\left[\frac{m+1}{2}f(\eta)+\frac{m-1}{2}\eta f'(\eta)\right]. \end{aligned} \tag{9}$$

In the present paper we are interested in the special case $m = -1$ of the velocity exponent in the *backward boundary layer* case $s = -1$, which corresponds to the fluid flow in a converging channel ("wedge nozzle") between two intersecting planes (see Schlichting and Gersten [25], Chap. 7.2.3; Fig. 7.4). In this case the free stream velocity (1) is

$$U(x) = -\frac{|U_\infty|\,L}{x}. \tag{10}$$

This velocity is directed toward the leading edge and corresponds to a potential sink.

The basic differential equation (7) and the velocity filed (9) reduce in this case to

$$f''' + 1 - f'^2 = 0, \tag{11}$$

$$u\,(x,y) = U(x)\,f'(\eta), \quad v\,(x,y) = \frac{U(x)}{\sqrt{Re}}\eta f'(\eta), \quad \eta = \sqrt{Re}\,\frac{y}{x}. \tag{12}$$

We mention that by the scale transformation $f \to a\,f$, $\eta \to a\,\eta$ with $a = \sqrt{2/(m+1)}$ Eq. (7) goes over into

$$sf''' + ff'' + \beta\left(1 - f'^2\right) = 0, \quad \beta = 2m/(m+1)\,, \tag{13}$$

which (for $s = +1$) is the more familiar form of the Fakner–Skan equation. This form, however, becomes singular for $m = -1$, and thus it is inadequate in the present application. As we are aware, in the *backward boundary layer* case $s = -1$, Eq. (13) has not been investigated until now for all m-values in detail.

3 Solution and discussion

It is easy to see that Eq. (11) admits the first integral

$$\frac{1}{2}\left[f''(\eta)\right]^2 + f'(\eta) - \frac{1}{3}\left[f'(\eta)\right]^3 = K_1. \tag{14}$$

Letting here $\eta \to \infty$, the asymptotic conditions yield for the constant of integration the value $K_1 = 2/3$. Furthermore, bearing in mind the boundary condition $f'(\infty) = \lambda$, one obtains for the similar wall shear stress $f''(0)$ the expression

$$f''(0) = \pm\,|\lambda - 1|\sqrt{\frac{2}{3}(\lambda + 2)}. \tag{15}$$

The wall shear stress $f''(0)$ as a function of the velocity ratio λ is plotted in Fig. 1.

Substituting $f'(\eta) = 6P(\eta)$ in Eq. (14), we obtain for $P(\eta)$ precisely the differential equation of Weierstrass' elliptic *P* function (see [2], Chap. 18)

$$P'^2 = 4P^3 - g_2 P - g_3 \tag{16}$$

with $g_2 = 1/3$ and $g_3 = -1/27$. Equation (16) is invariant under any translation $\eta \to \eta + \eta_0$ of the independent variable η. Thus, its general solution is of the form $P = P(\eta + \eta_0; g_2, g_3)$, where η_0 is the second constant of integration K_2 of the problem. Accordingly, the general solution for the similar velocity field is

$$f'(\eta) = 6P(\eta + \eta_0; g_2, g_3). \tag{17}$$

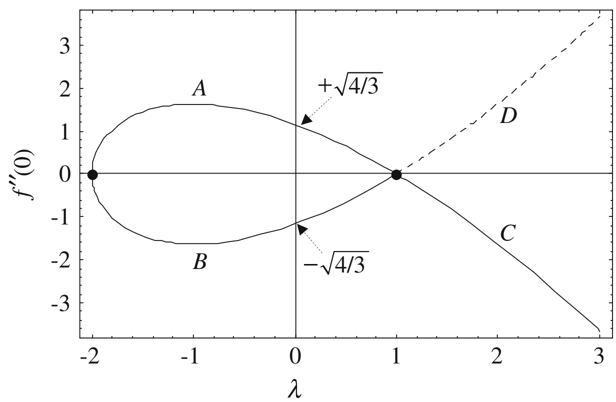

Fig. 1 Plot of the wall shear stress $f''(0)$ as a function of the velocity ratio λ. In the range $-2 < \lambda < 1$ dual solutions exist (*branches A* and *B*), while for $\lambda > 1$ the solution is unique (*branch C*). The solutions corresponding to the *branch D* are singular and thus, nonphysical

Taking into account that in present case the discriminant $\Delta = g_2^3 - 27g_3^2$ is vanishing, the P function can be expressed in terms of elementary hyperbolic functions as follows:

$$P\left(\eta + \eta_0; \frac{1}{3}, -\frac{1}{27}\right) = -\frac{1}{3} + \frac{1}{2}\tanh^2\left(\frac{\eta + \eta_0}{\sqrt{2}}\right). \tag{18}$$

Bearing in mind the boundary condition $f'(0) = \lambda$, the constant η_0 can be determined and the exact solutions of our boundary value problem becomes

$$f'(\eta) = -2 + 3\tanh^2\left(\frac{\eta}{\sqrt{2}} + \operatorname{arctanh}\sqrt{\frac{\lambda+2}{3}}\right) \quad \text{(branches A and C)}, \tag{19}$$

$$f'(\eta) = -2 + 3\tanh^2\left(\frac{\eta}{\sqrt{2}} - \operatorname{arctanh}\sqrt{\frac{\lambda+2}{3}}\right) \quad \text{(branch B)}. \tag{20}$$

Concerning the solution space of our boundary value problem (11), (8), a simple inspection of Fig. 1 leads to the following results:

1. Below the critical value $\lambda_c = -2$ of the velocity ratio λ, no solution exist.
2. In the range $-2 < \lambda < 1$, dual solutions with opposite signs of the wall shear stress $f''(0)$ occur (branches A and B)
3. At the lower bound $\lambda = \lambda_c = -2$ of the domain of existence, the dual solutions become coincident and the corresponding wall shear stress $f''(0)$ is zero.
4. For $\lambda = -1$, i.e. for equal but opposite wall and free stream velocities, $U_w(x) = -U(x)$, the wall shear stress $f''(0)$ reaches in the range $-2 < \lambda < 1$ of the dual solutions its extreme values $\pm\sqrt{8/3}$.
5. In the subintervals $-2 < \lambda < 0$, and $0 < \lambda < 1$, the dual solutions describe *opposing* and *aiding* flow regimes, respectively.
6. All the flows described by the B-branch solutions (20) involve backflow regions in the η-intervals

$$0 < \eta < \sqrt{2}\left(\operatorname{arctanh}\sqrt{\frac{2}{3}} + \operatorname{arctanh}\sqrt{\frac{\lambda+2}{3}}\right) \quad (-2 < \lambda < 1,\ \text{B branch}). \tag{21}$$

7. The flows described by the A-branch solutions (19) also involve backflow regions but only in the range $-2 < \lambda < 0$. The corresponding η-intervals are

$$0 < \eta < \sqrt{2}\left(\operatorname{arctanh}\sqrt{\frac{2}{3}} - \operatorname{arctanh}\sqrt{\frac{\lambda+2}{3}}\right) \quad (-2 < \lambda < 0,\ \text{A branch}). \tag{22}$$

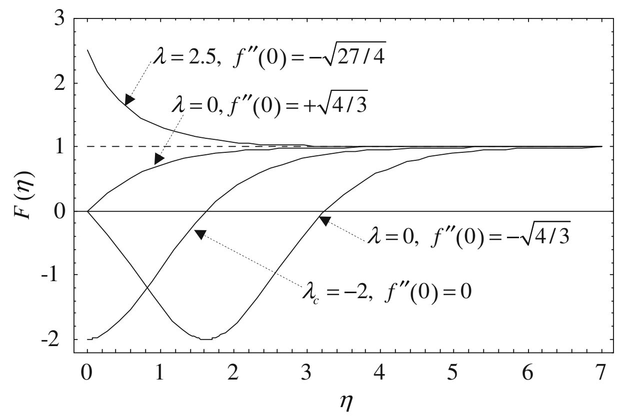

Fig. 2 Shown are four velocity profiles $F(\eta) = f'(\eta)$. Two of them correspond to the dual solutions for $\lambda = 0$. The other two are associated with the coincident dual solutions for $\lambda = \lambda_c = -2$, and with the unique solution for $\lambda = 2.5$, respectively

8. For $\lambda = 1$, i.e. for $U_w(x) = U(x)$, the dual solutions become coincident again. In this point of the parameter space $\{\lambda, f''(0)\}$, they reduce to the *universal solution* $f = \eta$ of the general boundary value problem (7), (8), regardless the sign function s, i.e. for *forward* and *backward* boundary layers as well.
9. In the range $\lambda > 1$ a unique (physical) solution exists for any given value of the velocity ratio λ (branch C). These solutions are given by Eq.(19) and describe *aiding* flow regimes. The solution (20), which is continued for $\lambda > 1$ in the branch D (dashed line), becomes singular and thus, it is nonphysical. An intuitive explanation of this (mathematical) phenomenon is given in Appendix A.
10. While in the range $-2 < \lambda < 1$ of the velocity ratio all the dual velocity profiles $f'(\eta)$ approach the asymptotic value $f'(\infty) = 1$ from below, the unique solutions $f'(\eta)$ in the aiding flow regime $\lambda > 1$ approach this value from above.

As an illustration of the above features in Fig. 2 four velocity profiles described by Eqs. (19) and (20) are shown. They correspond to $\lambda = \lambda_c = -2$ (coincident dual solutions with $f''(0) = 0$), $\lambda = 0$ (dual solutions with $f''(0) = \pm\sqrt{4/3}$ in the case of resting boundaries, $U_w = 0$) and $\lambda = 2.5$ (unique solution with $f''(0) = -\sqrt{27/4}$), respectively. For the parameter values $\{\lambda, f''(0)\} = \{0, +\sqrt{4/3}\}$ one recovers the famous Pohlhausen solution (without backflow region)

$$f'(\eta) = -2 + 3\tanh^2\left(\frac{\eta}{\sqrt{2}} + \operatorname{arctanh}\sqrt{\frac{2}{3}}\right), \quad f''(0) = +\sqrt{4/3}. \tag{23}$$

The solution (23) is the oldest known analytical closed form solution of Prandtl's boundary layer equations. It was reported by Karl Pohlhausen in its seminal ZAMM-paper [22] which has become later the basic reference for the celebrated Kármán–Pohlhausen integral method. On this historical background, it appears quite surprising that in the case of (23), the advantages offered by an exact solution (in investigation of the basic features of a theory) have not extensively been exploited during the development of boundary layer theory. The heat transfer characteristics of the flow (23) for a power law wall temperature distribution, e.g., have only very recently been considered [16]. Furthermore, as we are aware, the dual counterpart of (23),

$$f'(\eta) = -2 + 3\tanh^2\left(\frac{\eta}{\sqrt{2}} - \operatorname{arctanh}\sqrt{\frac{2}{3}}\right), \quad f''(0) = -\sqrt{4/3}, \tag{24}$$

which in the range $0 < \eta < 2\sqrt{2}\operatorname{arctanh}\left(\sqrt{2/3}\right) = 3.24199$ illustrates the basic phenomenon of flow inversion (backflow), has never been mentioned in the literature. The investigation of the heat transfer characteristics of this flow, as well as of the flows (19) and (20) in general, is still an open research opportunity.

At this place it is also worth discussing shortly the meaning of the asymptotic condition $u(x, \infty) = U(x)$ in the converging channel under consideration. Obviously, in this geometry the transversal coordinate y can not go to infinity at finite distances x from the origin $x=0$. As argued by Goldstein [9], in this case the boundary condition $u(x, \infty) = U(x)$ must be stated as

$$\lim_{Y\to\infty}\left[Y^N\left(\frac{u}{U} - 1\right)\right] = 0 \quad N(\text{real}), \tag{25}$$

where N is an arbitrary real number and

$$Y = \sqrt{Re}\, y \tag{26}$$

is the transversal coordinate y stretched by $\sqrt{Re}$ (in Goldstein's original paper, stretched by $\upsilon^{-1/2}$, i.e. $Y = y/\upsilon^{1/2}$). Since at the edge of the boundary layer y does not tend to infinity, in the asymptotic conditions the stretched coordinate Y must be used which goes to infinity for non-zero y when $\upsilon \to 0$ (i.e. for $Re \to \infty$, where the boundary layer approximation holds). In case of solutions (19) and (20) where $\eta = \sqrt{Re}\, y/x = Y/x$, the condition (25) becomes

$$\lim_{Y\to\infty}\left[Y^N f'\left(\frac{Y}{x}\right)\right] = -12 \lim_{Y\to\infty}\left[Y^N \exp\left(-\sqrt{2}\frac{Y}{x}\right)\right] = 0, \tag{27}$$

which is obviously satisfied for any real N.

4 A historical remark

The aim of this section is to amend the description given in the Introduction for the (generally adopted) historical development of the doubly-driven Blasius flows, just at its first cornerstone. Our remark concerns the existence of the dual solution in the range $\lambda_c = -0.3541 < \lambda < 0$ of the velocity ratio λ reported by Klemp and Acrivos in their Journal of Fluid Mechanics papers [13,14]. Well, a survey of the pertinent literature shows that exactly this result was reported few years earlier by J. Steinheuer [27]. Steinheuer uses in his paper a slightly different scaling (his asymptotic condition is $f'(\infty) = 2$), but the occurrence of the dual solutions in the λ-range $\lambda_c = -0.3541 < \lambda < 0$ is clearly seen in his Fig. 12. Although Steinheuer's paper was communicated by Hermann Schlichting and punblished in the *Transactions of the Scientific Society of Brunswick* (being also quoted in Schlichting and Gersten [25]), regrettably enough, his comprehensive investigation of the Blasius problem, and especially the above results, remained unnoticed by the broader fluid mechanics community. It is also worth noticing in this respect that very recently, the mentioned non-uniqueness occurring in the range $\lambda_c = -0.3541 < \lambda < 0$ was *rediscovered* once more by Ahmad [6], several decades after Steinheuer [27] and Klemp and Acrivos [13].

5 Summary and conclusions

The boundary layer flows in a converging channel (*wedge nozzle*) as a special case of the Falkner–Skan flows past stretching boundaries have been considered in this paper. Exact analytical solutions in terms of elementary hyperbolic functions were presented. The existence domain of the solutions in the parameter plane (λ, S) of the velocity ratio $\lambda = U_w(x)/U(x)$ and the wall shear stress $S = f''(0)$ has been investigated in detail. In the range $-2 < \lambda < +1$ dual solutions describing either opposing ($\lambda < 0$) or aiding ($\lambda > 0$) flow regimes were found. In the range $\lambda > 1$ unique solutions occur, while below the critical value $\lambda_c = -2$ no solutions exist at all. In the case $\lambda = 0$ of resting boundaries, Pohlhausen's classical solution and its dual counterpart were recovered. With the aid of a point mechanical analogy, the features of the solution space were explained intuitively. To the early history of the research field of Falkner–Skan flows past stretching boundaries a new relevant point has been added. The case of boundary layer flows in a converging channel examined in the present paper emphasizes the significance of exact analytical solutions in gaining of new knowledge once more.

Appendix A: A heuristic approach to the boundary value problem (11), (8)

The aim of this Appendix is to "explain" intuitively with the aid of a point mechanical analogy the structure of the domain of existence of solutions shown in Fig. 1. Namely, why the boundary value problem (11), (8) does not admit solutions for $\lambda < -2$, why dual solutions exist for $-2 < \lambda < 1$ (branches A and B), and why for $\lambda > 1$ only a single solution can exist (branch C), but the branch D is excluded. To this end, we first notice that the boundary value problem (11), (8) can be transcribed in the form

$$F'' = -\frac{\partial W}{\partial F}, \tag{A.1}$$

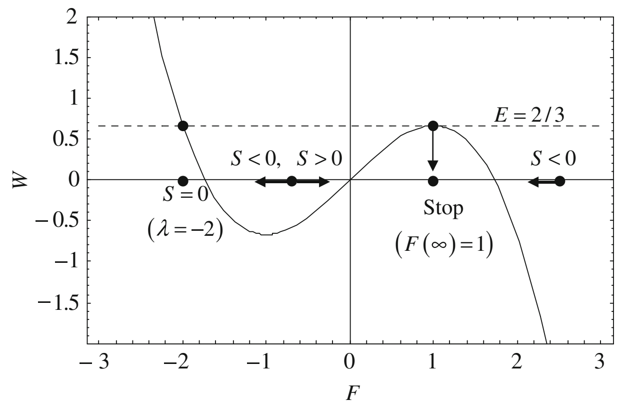

Fig. 3 Plot of the potential energy $S < 0$ of the analogous point mechanical problem. The *arrows* indicate the allowed directions of the initial velocity $S = F'(0)$ for different initial positions $\lambda = F(0)$ of the particle. The motion always stops in the point $F(\infty) = 1$ reached for $\eta \to \infty$

$$F(0) = \lambda \quad \text{and} \quad F(\infty) = 1 \tag{A.2}$$

where

$$F(\eta) = f'(\eta), \quad W = F - \frac{1}{3}F^3. \tag{A.3}$$

Thus, our boundary value problem can be reinterpreted as the point mechanical problem of a one dimensional motion. In this sense, Eq. (A.1) describes the conservative (i.e. frictionless) motion of a particle of coordinate F, mass $M = 1$ and potential energy W. The similarity variable η of the flow plays the role of time variable of the particle motion, the velocity ratio λ that of the initial position $\lambda = F(0)$and the wall shear stress $f''(0) = F'(0)$ that of the initial velocity of the particle. In Fig. 3 the shape of the potential energy W is shown. According to the first condition of (A.2) the particle is started at the "initial instant" $\eta = 0$ in the point of coordinate $F = \lambda$ of the F axis with a yet unknown "initial velocity" $F'(0) \equiv S$. According to the second condition (A.2) the particle ends its motion, as $\eta \to \infty$, in the point $F = 1$ with a vanishing terminal velocity $F'(\infty) = 0$. In this way, the existence and uniqueness problem of solutions of the boundary value problem (11), (8) reduces to the finding of all the initial velocities $F'(0) = S$ which guarantee that the particle started at the instant $\eta = 0$ in some point $F = \lambda$, ends its motion for $\eta \to \infty$ in the (mechanically unstable) equilibrium point $F = 1$ corresponding to the maximum value of $W_{\max} = 2/3$ of the potential energy W.

The first benefit from this point mechanical analogy is that the first integral (14) can be recovered without any mathematical effort as the conservation law of the mechanical energy of the particle,

$$\frac{1}{2}F'^2 + W = \text{const.} \equiv E \tag{A.4}$$

The requirement that the motion stops in the equilibrium point $F(\infty) = 1$ implies that the (conserved) mechanical energy of the particle is $E = 2/3$. Consequently, the energy conservation (A.4) and the initial condition $F(0) = \lambda$ imply that the initial velocity of the particle, $F'(0) = S$, as a function of its initial position λ must be chosen as

$$S = \pm\sqrt{2\left(E - W|_{F=\lambda}\right)} = \pm\sqrt{\frac{2}{3}\left(2 - 3\lambda + \lambda^3\right)} = \pm\,|\lambda - 1|\sqrt{\frac{2}{3}\,(2 + \lambda)}. \tag{A.5}$$

This equation coincides precisely with Eq. (15).

The allowed domain of the particle motion consists of that range of the F-axis for which the inequality

$$W(F) \leq E = 2/3 \tag{A.6}$$

holds. Out of this domain the kinetic energy $F'^2/2$ would be negative which would require an imaginary particle velocity. Thus, one immediately sees that the domain accessible for the particle is the range $F \geq -2$, $F = -2$ being a turning point of motion. In other words, below of $F = -2$ no initial position $\lambda = F(0)$ can exist. In order to stop at $F = 1$ as $\eta \to \infty$, the particle with initial position $\lambda = F(0) = -2$, must be started with vanishing initial velocity $S = 0$. The corresponding solution is unique. When the initial position $\lambda = F(0)$ is somewhere the turning point $F = -2$ and the abscissa $F = +1$ corresponding to the maximum of the potential energy W (i.e. when $-2 < \lambda < 1$), the particle can be started either to the right or to the left with the positive or negative initial velocity S given by the respective Eqs. (A.5). The particle motion stops in the both cases in $F = 1$ as $\eta \to \infty$. The corresponding laws of the particle motion are the dual solutions of the fluid mechanical problem. When the initial position of the particle coincides with the equilibrium point $F(\infty) = 1$ where, as a requirement of the energy conservation, its initial velocity S must be zero, the particle remains in this point for all times $\eta > 0$. Furthermore, when the initial position λ is to the right of the equilibrium point $F = 1$, the particle can reach the asymptotic state $F(\infty) = 1$ only with a negative initial velocity S. This motion corresponds to the unique solution of the flow problem for $\lambda > 1$. In the opposite case, of a positive S (and of an initial position $\lambda > 1$), the particle motion becomes necessarily unbounded. This motion of the particle corresponds to the singular, i.e. non physical solution of the flow problem (branch D). Therefore, the above point mechanical analogy shows intuitively that reason for the inexistence of solutions with ($\lambda > 1$, $S > 0$) is a topological one.

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